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Non-stationary two-dimensional potential flows by the Broadwell model equations

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Abstract – The two-dimensional Broadwell model of discrete kinetic theory is studied in order to clarify the physical relevance of its solutions in comparison to the solutions of the continuous Boltzmann equation. This is achieved by determining completely, in closed form, all non-stationary potential flows with steady limiting conditions and isotropic pressure tensor at infinity. Several classes of exact solutions are also constructed when some of the above hypotheses are dropped. Most results are made possible by suitable transformations, which reduce essentially a complicated overdetermined system of partial differential equations to solving explicitly a Liouville equation. The structure of the obtained solutions, and especially the unphysical features that they exhibit, are finally commented on. It is remarkable that, for the problem considered here, there is no solution showing the typical qualitative features which characterize the continuous Boltzmann equation. © 2000 Éditions scientifiques et médicales Elsevier SAS

kinetic theory / discrete velocity models / potential flows

1. Introduction

The paper is devoted to the investigation of the classical two-dimensional Broadwell model [1] of the Boltzmann equation. The model was previously studied by many authors from various view points. One important question is whether or not the solutions to the model are qualitatively close to solutions of the Boltzmann equation for physically reasonable problems, in spite of the oversimplification of the model itself. In other words: is the Broadwell model simply an interesting (mathematically) quasilinear hyperbolic system, or, instead, can the model really help in a deeper understanding of the Boltzmann equation?

The question is important also in the allied fields of Lattice Gas Cellular Automata and Lattice Boltzmann Equation, although space is discrete in those applications. In such models in fact, the use of very simple collision terms, with only a few velocities, implies a quite large probability that the obtained numerical solutions are unphysical, even though sensible hydrodynamic results might be in order in some physical situations. In addition, the N-particle Lattice Gas model on a square lattice, studied in [2], leads directly, under the assumption of molecular chaos, to the present nonlinear discrete 4-velocity Broadwell equations. It is well known a priori, that the macroscopic equations of the model, with its square symmetry, differ quantitatively from the hydrodynamic equations of an isotropic fluid, and that the symmetry of the allowed velocity set should be hexagonal/triangular in two dimensions, in order to have a viscosity tensor with the proper symmetry. However, this does not contradict the possible usefulness of the Broadwell equations as a purely kinetic model, provided its solutions are qualitatively similar to the ones of the Boltzmann equation.

We try to clarify the above question on the basis of the explicit analytical solution to a given problem. Many exact solutions of the Broadwell model were previously constructed in several pioneering papers by Cornille

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(see, for example, [3,4]). Our aim, however, is to consider the problem in a more systematic way. In a previous paper [5] a class of potential flows for the stationary two-dimensional Broadwell model was completely described; moreover, an interesting new class of stationary solutions was recently found by Cabannes [6]. We try below to make a further step, after [5], by generalizing some ideas used there to the non-stationary case. In particular, we construct below the complete class of non-stationary potential flows under certain conditions at infinity.

The paper is organized as follows. We consider the 4-velocity Broadwell model in Section 2, and reduce it to a system of three equations for new unknown functions. Then, we define a class of steady vorticity solutions, and show that such solutions are described by a system of two equations for two unknowns $\varphi(x, y, t)$ (potential) and $\psi(x, y)$ (vorticity). The unsteady potential flows ($\psi \equiv 0$) are considered in detail in Sections 3–6. To describe the whole class of such flows we need to find all solutions of two equations for a single function of x, y, and t (overdetermined system). In Sections 3–4 we solve the problem under the additional condition that the stress tensor of the flow at infinity is isotropic, and prove that there exist only 7 independent classes of such flows. These classes of solutions are given in explicit form in Section 5, which collects the main results of the present paper. Some of solutions are new, whereas others coincide with the known Cornille's solutions. We stress that finding the solutions is much easier than proving that they indeed exhaust all the above described potential flows. The proof needs rather long calculations, and is based on the fact that the general solution of the Liouville equation $u_{xy} = \mathrm{e}^u$ is known in explicit form. Some generalizations are considered in Section 6. Finally, we analyze the solutions, and discuss their non-physical properties in comparison with the properties of the solutions of the Boltzmann equation. The comparison gives the desired quantitative evidence of the unreliability of this simple model, which, for the considered problem, does not provide any solution with the qualitative properties that are expected from the Boltzmann equation.

2. The Broadwell model equations

We consider the well known plane dimensionless Broadwell model equations for distribution functions $f_i(x, y, t),\ i = 1, \ldots, 4$, of particles with velocities $\mathbf{v}_1 = (1, 0),\ \mathbf{v}_2 = (0, 1),\ \mathbf{v}_3 = (-1, 0)$ and $\mathbf{v}_4 = (0, -1)$, respectively. The equations are [1,7]

$$\frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x} = -\frac{\partial f_2}{\partial t} - \frac{\partial f_2}{\partial y} = \frac{\partial f_3}{\partial t} - \frac{\partial f_3}{\partial x} = -\frac{\partial f_4}{\partial t} + \frac{\partial f_4}{\partial y} = f_2 f_4 - f_1 f_3. \tag{1}$$

Introduce the drift velocity $\mathbf{u} = (u_1, u_2)$ with

$$u_1 = f_1 - f_3, \qquad u_2 = f_2 - f_4. \tag{2}$$

Notice that the stress tensor, proportional to $\langle \mathbf{vv} \rangle$, is always diagonal here since $v_{ix} v_{iy} = 0$, $\forall i = 1, 2, 3, 4$. Introduce then its diagonal components

$$p_1 = f_1 + f_3, \qquad p_2 = f_2 + f_4. \tag{3}$$

In the new unknowns the governing equations split as

$$\frac{\partial u_1}{\partial t} + \frac{\partial p_1}{\partial x} = 0, \qquad \frac{\partial u_2}{\partial t} + \frac{\partial p_2}{\partial y} = 0 \tag{4}$$

and

$$\frac{\partial p_1}{\partial t}+\frac{\partial p_2}{\partial t}+\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}=0, \tag{5}$$

which are linear and of (macroscopic) conservation type, plus the nonlinear equation of kinetic type

$$\frac{\partial p_1}{\partial t}-\frac{\partial p_2}{\partial t}+\frac{\partial u_1}{\partial x}-\frac{\partial u_2}{\partial y}=p_2^2-p_1^2-u_2^2+u_1^2. \tag{6}$$

Any smooth two-dimensional vector, like $\mathbf{u}$, may be written as

$$u_1=\varphi_x+\psi_y, \qquad u_2=\varphi_y-\psi_x, \tag{7}$$

where the scalar fields φ and ψ are such that $\nabla^2\varphi$ and $-\nabla^2\psi$ represent respectively divergence and curl of the vector $\mathbf{u}$. It is then easily realized from (4), (5) that the vectors (φ_t+p_1,ψ_t) and $(-\psi_t,\varphi_t+p_2)$ are divergence free, and then it is not difficult to prove that there exists a smooth scalar field ϕ such that

$$p_1=-\varphi_t+\phi_{yy}, \qquad p_2=-\varphi_t-\phi_{xx}, \tag{8}$$

which, together with (7), allows us to determine all unknowns in terms of φ, ψ, and ϕ. The relevant equations take the form

$$\begin{gathered}\psi_t+\phi_{xy}=0,\\ 2\varphi_{tt}+\frac{\partial}{\partial t}\left(\phi_{xx}-\phi_{yy}\right)=\nabla^2\varphi,\\ \nabla^2\phi_t+\varphi_{xx}-\varphi_{yy}+2\psi_{xy}=\phi_{xx}^2-\phi_{yy}^2+2\varphi_t\nabla^2\phi+(\varphi_x+\psi_y)^2-(\varphi_y-\psi_x)^2,\end{gathered} \tag{9}$$

and the particle distribution function is recovered as

$$\begin{aligned} f_1&=\frac{1}{2}\left(\phi_{yy}-\varphi_t+\varphi_x+\psi_y\right), \qquad & f_2&=\frac{1}{2}\left(-\phi_{xx}-\varphi_t+\varphi_y-\psi_x\right),\\ f_3&=\frac{1}{2}\left(\phi_{yy}-\varphi_t-\varphi_x-\psi_y\right), \qquad & f_4&=\frac{1}{2}\left(-\phi_{xx}-\varphi_t-\varphi_y+\psi_x\right). \end{aligned} \tag{10}$$

The above transformations and equations involve only derivatives of φ, ψ, and ϕ, and such functions are of course not uniquely defined. It is clear that everything remains unchanged upon addition to φ and ψ of two conjugate harmonic functions, Φ and Ψ respectively, and to ϕ of any function of the kind $A(t)x+B(t)y+C(t)$. Another noticeable invariance property is that we are free to add to each of φ and ψ an arbitrary smooth function of time, say α and β, respectively, leaving everything unchanged provided

$$\frac{1}{2}\alpha'(t)\left(y^2-x^2\right)-\beta'(t)xy$$

is added to ϕ, where a prime denotes derivative.

We shall consider in the following solutions to (9) for which ψ, which defines the vorticity, does not depend on time. This implies at once

$$\phi(x,y,t)=a(x,t)+b(y,t), \tag{11}$$

with a and b arbitrary functions to be determined. Let us also introduce a class of boundary conditions which correspond to steady flow at infinity, namely

$$p_i \to p_i^\infty = \text{constant}, \quad \text{if} \quad x^2 + y^2 \to \infty, \quad i = 1, 2, \tag{12}$$

and set

$$\rho_\infty = p_1^\infty + p_2^\infty, \qquad q_\infty = p_1^\infty - p_2^\infty. \tag{13}$$

Then, according to (8), $a_{xx} + b_{yy} \to q_\infty$ if $x^2 + y^2 \to \infty$, and therefore, since a is independent of y and b is independent of x,

$$a_{xx} = A(t), \qquad b_{yy} = B(t), \quad A(t) + B(t) = q_\infty. \tag{14}$$

Integrating and getting rid of inessential terms leads to

$$\phi(x, y, t) = \frac{1}{2}A(t)x^2 + \frac{1}{2}B(t)y^2,$$

where we may always set $-A(t) + q_\infty/2 = B(t) - q_\infty/2 = \alpha'(t)$, so that, resorting to the last invariance property quoted above, we may reduce it to the simple stationary and symmetric expression

$$\phi = \frac{1}{4}q_\infty\left(x^2 + y^2\right). \tag{15}$$

For the unknowns $\varphi(x, y, t)$ and $\psi(x, y)$ we are thus left with the pair of coupled equations

$$\varphi_{tt} = \frac{1}{2}\nabla^2\varphi, \qquad \varphi_{xx} - \varphi_{yy} + 2\psi_{xy} = (\varphi_x + \psi_y)^2 - (\varphi_y - \psi_x)^2 + 2q_\infty\varphi_t. \tag{16}$$

When the condition $\psi_t = 0$ is replaced by the stronger $\psi = 0$, which corresponds to zero vorticity, and then to potential flow, one is left with an overdetermined set of two equations for a single unknown φ. The simplest case to be studied is obviously when we assume in addition $q_\infty = 0$, which corresponds to a flow at infinity with an isotropic pressure tensor. Such a case will be examined in detail in the next three sections. The more general potential flow with $q_\infty \neq 0$ will be briefly discussed in Section 6.

3. Unsteady potential flows

With $\psi = 0,\ q_\infty = 0$, and condition (12) at infinity, we have to solve for the unknown φ the overdetermined set of PDE's in three independent variables

$$\varphi_{tt} = \frac{1}{2}\nabla^2\varphi, \qquad \varphi_{xx} - \varphi_{yy} = \varphi_x^2 - \varphi_y^2. \tag{17}$$

It is clear that φ is defined only up to an additive constant, and that the whole problem is invariant under exchange of x with y, of x with $-x$, and of y with $-y$, so that any solution $\varphi(x, y, t)$ implies existence of the other solutions $\varphi(y, x, t),\ \varphi(-x, y, t),\ \varphi(x, -y, t)$, etc. Introducing the substitution

$$x + y = \tilde{x}, \qquad x - y = \tilde{y}, \qquad \varphi\left(\frac{\tilde{x} + \tilde{y}}{2}, \frac{\tilde{x} - \tilde{y}}{2}, t\right) = \tilde{\varphi}(\tilde{x}, \tilde{y}, t) \tag{18}$$

and eliminating all tildes yields

$$\varphi_{tt} = \nabla^2\varphi, \qquad \frac{\partial^2}{\partial x \partial y}\left(e^{-\varphi}\right) = 0, \tag{19}$$

where again x and y may be interchanged. The second is solved by $\varphi = -\log\omega$, with $\omega(x, y, t) = u(x, t) + v(y, t)$, u and v arbitrary, and from the first we have

$$\omega(\omega_{tt} - \omega_{xx} - \omega_{yy}) = \omega_t^2 - \omega_x^2 - \omega_y^2. \tag{20}$$

Since $\omega = u + v$ is the only quantity of interest, and everything remains unchanged upon addition of $g(t)$ and $-g(t)$ to u and v respectively, g being any function of t alone, the function u (and, conversely, v) is always defined up to an additive function of t.

If either u or v vanishes, the other follows easily. For instance, for $v = 0$, we have $u = e^{-\varphi}$, $\varphi_{tt} - \varphi_{xx} = 0$, and then

$$\varphi(x, y, t) = \Phi(x + t) + \Psi(x - t), \tag{21}$$

where Φ and Ψ are smooth arbitrary functions of one real variable. If, conversely, $u = 0$, one would get correspondingly the other special solution

$$\varphi(x, y, t) = \Phi(y + t) + \Psi(y - t), \tag{22}$$

again with arbitrary Φ and Ψ. These are indeed the same solutions as in (21), in the sense of the established invariance properties.

Coming to the general case of u and v both nonvanishing (i.e., u_x and v_y both nonzero), equation (20) may be cast as the equality of a function of x, y, t to the sum of a function of x and t with a function of y and t. Introducing $U = u_x$ and $V = v_y$, careful and lengthy manipulations, omitted for brevity, based on proper compatibility conditions and invariance properties, show that U is separable as $U(x, t) = W(x)T(t)$, and that $W''/W = k^2$, where k^2 is a real constant. Therefore we end up with the form $u(x, t) = q(x)T(t)$, where q has different expressions according to whether $k^2 > 0$ ($\sqrt{k^2} = k > 0$), $k^2 = 0$, or $k^2 < 0$ ($\sqrt{k^2} = i\kappa, \kappa > 0$). More precisely

$$\begin{aligned} q(x) &= c_1 e^{kx} + c_2 e^{-kx}, \quad k^2 > 0, \\ q(x) &= c_1 x^2 + c_2 x, \quad k^2 = 0, \\ q(x) &= c_1 e^{i\kappa x} + c_2 e^{-i\kappa x}, \quad k^2 < 0, \end{aligned} \tag{23}$$

with c_1 and c_2 integration constants, real for $k^2 \geqslant 0$, complex conjugate for $k^2 < 0$, so that the product c_1c_2 is always positive in the latter case, except for the trivial one $c_1 = c_2 = 0$. Consider first the case $k \neq 0$, where u is determined apart from an arbitrary function $T(t)$ and three scalar parameters c_1, c_2, k. After entering such a result back into (20), further manipulations lead first to $T''T - T'^2 = 0$, which determines the function T as $T(t) = e^{\lambda t}$ (λ arbitrary real constant), and yield then the overdetermined set of PDE's in two independent variables for v

$$\begin{cases} v_{tt} - v_{yy} - 2\lambda v_t + \left(\lambda^2 - k^2\right)v = 0, \\ v(v_{tt} - v_{yy}) - v_t^2 + v_y^2 = 4c_1c_2k^2e^{2\lambda t}. \end{cases} \tag{24}$$

The function u is determined up to four scalar pameters as

$$u(x, t) = q(x)e^{\lambda t}. \tag{25}$$

When $k=0$, we may take $c_2\geqslant 0$ in (23), since (20) does not change if x is replaced by $-x$. The same machinery as before applies, and yields, after some algebra,

$$\begin{cases} v_{tt}-v_{yy}-2\lambda v_t+\lambda^2 v=-2c_1\mathrm{e}^{\lambda t},\\ v(v_{tt}-v_{yy})-{v_t}^2+{v_y}^2-2c_1\mathrm{e}^{\lambda t}v=-c_2^2\mathrm{e}^{2\lambda t}, \end{cases} \tag{26}$$

with u determined up to three scalar parameters again by (25) and (23).

4. Reduction to the Liouville equation

If k^2 is positive and $c_1c_2\neq 0$, we may set in (24)

$$v(y,t)=2\sqrt{|c_1c_2|}\,w(ky,kt)\mathrm{e}^{\lambda t} \tag{27}$$

to get for the new unknown $w(y,t)$, in terms of $\xi=(y+t)/2$ and $\eta=\operatorname{sgn}(c_1c_2)(y-t)/2$, the reduced set of equations

$$w_{\xi\eta}+\operatorname{sgn}(c_1c_2)w=0,\qquad w^2(\log w)_{\xi\eta}+1=0. \tag{28}$$

If k^2 is negative (then $c_1c_2>0$), we put instead

$$v(y,t)=2\sqrt{c_1c_2}\,w(\kappa y,\kappa t)\mathrm{e}^{\lambda t} \tag{29}$$

and, with $\xi=(y+t)/2$ and $\eta=-(y-t)/2$, we obtain again (28), with $\operatorname{sgn}(c_1c_2)$ simply replaced by 1.

If $k\neq 0$ and $c_1c_2=0$, for

$$w(y,t)=v(y,t)\mathrm{e}^{-\lambda t} \tag{30}$$

the second equation in (24) is homogeneous and its general integral reads as

$$w(y,t)=X(y+t)Y(y-t). \tag{31}$$

The arbitrary functions X and Y are determined by the first equation as

$$X=d_1\mathrm{e}^{\varepsilon(y+t)},\qquad Y=d_2\mathrm{e}^{-\frac{k^2}{4\varepsilon}(y-t)}, \tag{32}$$

where d_1, d_2 and $\varepsilon\neq 0$ are arbitrary real constants.

If $k=0$ and $c_1=0$ (then $c_2>0$), we set in (26)

$$v(y,t)=w(c_2y,c_2t)\mathrm{e}^{\lambda t} \tag{33}$$

and get for $w(y,t)$

$$w_{tt}-w_{yy}=0,\qquad w_t^2-w_y^2=1 \tag{34}$$

and then, by simple calculations,

$$w(y,t)=\varepsilon(y+t)-\frac{1}{4\varepsilon}(y-t)+d, \tag{35}$$

where $\varepsilon \neq 0$ and d are arbitrary real constants.

If $k = 0$, $c_1 \neq 0$ and $c_2 > 0$, from (33) we get

$$w_{tt} - w_{yy} = -2\frac{c_1}{{c_2}^2}, \qquad w_t^2 - w_y^2 + 4\frac{c_1}{{c_2}^2}w = 1 \tag{36}$$

and the general integral of the linear equation reads as

$$w(y,t) = X(y+t) + Y(y-t) + \frac{c_1}{2{c_2}^2}(y^2 - t^2).$$

Inserting this into the nonlinear one yields, after some algebra, with $\xi = y + t$ and $\eta = y - t$,

$$X = \alpha\xi^2 + d_1\xi + d_2, \qquad Y = \frac{{c_1}^2}{16\alpha {c_2}^4}\eta^2 + d_3\eta + d_4,$$
$$d_3 = \frac{c_1}{4\alpha {c_2}^2}d_1, \qquad d_4 = \frac{{c_2}^2}{4c_1} + \frac{{d_1}^2}{4\alpha} - d_2$$

with $\alpha > 0$, d_1, and d_2 arbitrary constants. Thus, upon renaming constants, we end up with

$$v(y,t) = \mathrm{e}^{\lambda t}\left\{\alpha\left[y + t + \frac{c_1}{4\alpha}(y-t) + \frac{\beta}{2\alpha}\right]^2 + \frac{{c_2}^2}{4c_1}\right\}, \tag{37}$$

with $\alpha \neq 0$, β, $c_1 \neq 0$, $c_2 > 0$ arbitrary real parameters.

In the case $k = 0$, $c_1 \neq 0$, $c_2 = 0$, it is not difficult to show (details are omitted) that v is just given by the specialization of (37) to the limiting case $c_2 = 0$.

For a general $k^2 \neq 0$ and $c_1c_2 \neq 0$ one has to solve (28). The substitution $w = \sqrt{2}\exp(-z/2)$ transforms the second of them (the nonlinear one) into the Liouville equation

$$z_{\xi\eta} = \mathrm{e}^z, \tag{38}$$

whose general integral is explicitly known as [8]

$$z(\xi,\eta) = \log\left\{\frac{2a'(\xi)b'(\eta)}{[a(\xi) + b(\eta)]^2}\right\}$$

with a and b arbitrary functions, $a'b' > 0$. Thus

$$w(\xi,\eta) = \pm\frac{a(\xi) + b(\eta)}{\sqrt{a'(\xi)b'(\eta)}}, \tag{39}$$

where we may always assume $a' > 0$ and $b' > 0$ because of the possibility of exchanging a in $-a$ and simultaneously b in $-b$. With reference to the plus sign in (39), substitution into the first of (28) (the linear one) yields a compatibility condition which looks quite awkward, but can be properly handled to yield, after patient manipulations, and upon introducing arbitrary constants μ, $d_1 \neq 0$, and $d_2 \neq 0$, the set of coupled equations

$$a' = \frac{1}{d_2}\operatorname{sgn}(c_1c_2)(a - \mu)^2 + d_1, \qquad b' = \frac{1}{d_1}\operatorname{sgn}(c_1c_2)(b + \mu)^2 + d_2. \tag{40}$$

Moreover, since the transformation $a \to a - \mu,\ b \to b + \mu$ does not change w in (39), we may take $\mu = 0$ without loss of generality. If now $d_1 d_2 > 0$, the substitution

$$a(\xi) = \sqrt{d_1 d_2}\, a_1\left(\sqrt{\frac{d_1}{d_2}}\xi\right), \qquad b(\eta) = \sqrt{d_1 d_2}\, b_1\left(\sqrt{\frac{d_2}{d_1}}\eta\right) \tag{41}$$

reduces the ordinary differential equations for a and b to

$$a_1{}'(\xi) = 1 + \operatorname{sgn}(c_1 c_2) a_1^2(\xi), \qquad b_1{}'(\eta) = 1 + \operatorname{sgn}(c_1 c_2) b_1^2(\eta),$$

which can be explicitly solved. If $c_1 c_2 > 0$ we have

$$a_1(\xi) = \tan(\xi + \gamma_1), \qquad b_1(\eta) = \tan(\eta + \gamma_2), \tag{42}$$

where γ_1 and γ_2 are integration constants, whereas we get

$$a_1(\xi) = \tanh(\xi + \gamma_1), \qquad b_1(\eta) = \tanh(\eta + \gamma_2), \tag{43}$$

for $c_1 c_2 < 0$. In the other case $d_1 d_2 < 0$ we may always assume $d_1 > 0$, $d_2 < 0$, since the substitution $\xi \to -\xi,\ \eta \to -\eta$ leaves (28) invariant. The previous substitution is replaced by

$$a(\xi) = \sqrt{-d_1 d_2}\, a_1\left(\sqrt{-\frac{d_1}{d_2}}\xi\right), \qquad b(\eta) = \sqrt{-d_1 d_2}\, b_1\left(\sqrt{-\frac{d_2}{d_1}}\eta\right),$$

which yields

$$a_1'(\xi) = 1 - \operatorname{sgn}(c_1 c_2) a_1^2(\xi), \qquad b_1'(\eta) = -1 + \operatorname{sgn}(c_1 c_2) b_1^2(\eta).$$

The explicit solutions are easily seen to always violate the constraint $a'b' > 0$, and therefore the option $d_1 d_2 < 0$ must be discarded.

Now one has to go all the way back in order to write down the solutions of (17): it is remarkable that it has been possible to determine all of them analytically. The distribution function is then recovered via (10) with $\phi = \psi = 0$.

5. Admissible solutions

The previous long investigation has provided all solutions to the Broadwell model equations for potential flows subject to boundary conditions (12) with $q_\infty = 0$. We list and discuss here all of them after rewriting in terms of the original independent variables, re-arranging integration constants, when convenient, and getting rid of inessential constants that would not affect the f_i.

The solutions can be grouped in the following classes.

$$\varphi_1(x, y, t) = \Phi(x + y + t) + \Psi(x + y - t), \tag{44}$$

where Φ and Ψ are arbitrary smooth functions.

$$\varphi_2 = -\lambda t - \log\left[x + y + \frac{4\theta^2 - 1}{4\theta}\left(x - y + \frac{4\theta^2 + 1}{4\theta^2 - 1}t\right) + \gamma\right] \tag{45}$$

depending on three real parameters λ, θ, γ, with $\theta \neq 0$. Notice that the modulus of the speed of propagation, $(4\theta^2+1)/(4\theta^2-1)$, is always greater than unity.

$$\varphi_3 = -\lambda t - \log\left\{\left(x+y+\frac{\gamma}{2}\right)^2 + \beta\left[\frac{4\beta+1}{4\beta}\left(x-y+\frac{4\beta-1}{4\beta+1}t\right)+\frac{\delta}{2\beta}\right]^2\right\}, \tag{46}$$

depending on four real parameters $\lambda, \beta, \gamma, \delta$, with $\beta \neq 0$. The modulus of the propagation speed is smaller than unity for $\beta > 0$, and greater than unity for $\beta < 0$.

$$\varphi_4 = -\lambda t - \log\left\{\exp\left[k(x+y)\right] + \beta\exp\left[\frac{4\theta^2-k^2}{4\theta}\left(x-y+\frac{4\theta^2+k^2}{4\theta^2-k^2}t\right)\right]\right\}, \tag{47}$$

which depends on four real parameters $\lambda, k, \theta, \beta$, with $k \neq 0$ and $\theta \neq 0$. Here the modulus of the propagation speed is greater than unity since k^2 is positive. Notice that φ_4 still makes sense for $k = 0$, but becomes then a particular case of φ_1.

$$\varphi_5 = -\lambda t - \log\left\{\cosh\left[k(x+y)+\gamma\right] + \cos\left[\frac{k}{2}\frac{\theta^2+1}{\theta}\left(x-y+\frac{\theta^2-1}{\theta^2+1}t\right)+\delta\right]\right\}, \tag{48}$$

which depends on five real parameters $\lambda, k, \theta, \gamma, \delta$, with $k > 0$ and $\theta \neq 0$. Here $(\theta^2-1)/(\theta^2+1)$ is always less than one in modulus.

$$\varphi_6 = -\lambda t - \log\left\{\sinh\left[k(x+y)+\gamma\right] + \sinh\left[\frac{k}{2}\frac{\theta^2-1}{\theta}\left(x-y+\frac{\theta^2+1}{\theta^2-1}t\right)+\delta\right]\right\}, \tag{49}$$

depending again on five real parameters $\lambda, k, \theta, \gamma, \delta$, with $k > 0$ and $\theta \neq 0$. Here instead $(\theta^2+1)/(\theta^2-1)$ is always larger than one in modulus.

$$\varphi_7 = -\lambda t - \log\left\{\sin\left[\kappa(x+y)+\gamma\right] + \beta\sin\left[\frac{\kappa}{2}\frac{\theta^2-1}{\theta}\left(x-y+\frac{\theta^2+1}{\theta^2-1}t\right)+\delta\right]\right\}, \tag{50}$$

depending on six real parameters $\lambda, \kappa, \theta, \beta, \gamma, \delta$, with $\kappa > 0, \theta \neq 0$ and $0 < \beta < 1$.

According to the established invariance properties, several more classes are obtained by exchanging x and $-x$, y and $-y$, x and y, and combinations thereof, which all amounts essentially only to suitable rotations in the (x, y) plane. We refer below to the seven classes above and formulate the result proved in Sections 3 and 4.

PROPOSITION: *All solutions of* (9) *with* $\psi = \phi = 0$ *satisfying condition* (12) *reduce to the seven classes* (44)–(50), *plus their invariant* (*with respect to* (9)) *transformations.*

Remark 1: A key point of the proof is that the general solution of the Liouville equation (38) is known in explicit form. Finding the solutions (44)–(50) would be much easier than proving that there are no other solutions of (9) with $\psi = \phi = 0$.

Remark 2: All above solutions describe travelling (or periodic) waves directed along one of the diagonals $x = \pm y$. The directions are related to the assumption $\phi = 0$. Then $p_1 = p_2$ in (8), and therefore a steady (equilibrium) solution at infinity is such that $|u_1^\infty| = |u_2^\infty|$, according to (6). Thus, a direction of the drift velocity $\mathbf{u}^\infty = (u_1^\infty, u_2^\infty)$ at infinity defines a direction of the wave propagation.

The distribution function is evaluated now on using (10) with $\psi = \phi = 0$. It is clear that $\varphi_1(x, y, t)$ represents waves travelling at the particle speed along one diagonal in both directions. They correspond to collisionless solutions, in the sense that $f_1 = f_2$ and $f_3 = f_4$, so that the collision terms in (1) all vanish. Of course, the same would apply to $\varphi_1(x, -y, t)$ along the other diagonal, with $f_1 = f_4$ and $f_2 = f_3$. In all other φ_n's we have appearance only of functions with arguments $x \pm y - \zeta t$, with a propagation speed ζ which either is zero, or depends on the allowed free parameters. The classes described by $\varphi_2,\ \varphi_3$ with $\beta < 0$, $\varphi_4,\ \varphi_6$, and φ_7 appear then of questionable physical meaning, since they exhibit a speed of propagation greater than the particle speed. In all cases, positivity of the distribution function on the whole (x, y) plane can be guaranteed in suitable regions of nonzero measure in parameter space, in particular, on playing on λ. The only exception is when the content of the curly brackets in the formulas above may vanish, in which case positivity can be obtained only apart from arbitrarily small neighbourhoods of a discrete set of singular points, representing movable sources or sinks for the flow [5].

6. Generalization to the case $q_\infty \neq 0$

In this section we shall comment briefly on the existence of particular exact solutions for potential flows with boundary conditions (12) and $q_\infty \neq 0$, namely non-isotropic stress tensor at infinity. We shall deal with

$$\varphi_{tt} = \frac{1}{2}\nabla^2\varphi, \qquad \varphi_{xx} - \varphi_{yy} = \varphi_x^2 - \varphi_y^2 + 2q_\infty\varphi_t \tag{51}$$

and recover the distribution function via (10) with $\psi = 0$ and ϕ given by (15). Introducing

$$\tilde{x} = q_\infty\frac{x+y}{2},\ \tilde{y} = q_\infty\frac{x-y}{2},\ \tilde{t} = q_\infty\frac{t}{2},\ \varphi\left(\frac{\tilde{x}+\tilde{y}}{q_\infty}, \frac{\tilde{x}-\tilde{y}}{q_\infty}, \frac{2\tilde{t}}{q_\infty}\right) = \tilde{\varphi}(\tilde{x}, \tilde{y}, \tilde{t}) \tag{52}$$

and dropping all tildes leads to

$$\varphi_{tt} = \nabla^2\varphi, \qquad \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x\partial y}\right)\mathrm{e}^{-\varphi} = 0. \tag{53}$$

Let us look for solutions of the same kind as for $q_\infty = 0$, namely

$$\varphi = \hat{\varphi}(\xi, \eta) + at, \quad \xi = x - ct,\ \eta = y, \tag{54}$$

assuming $|c| < 1$. The further substitution

$$\hat{\varphi}(\xi, \eta) = c\eta + \hat{\psi}(p, q), \quad p = \frac{\xi}{\sqrt{1-c^2}},\ q = \eta, \tag{55}$$

yields

$$\hat{\psi}_{pp} + \hat{\psi}_{qq} = 0, \qquad \hat{\psi}_{pq} - \hat{\psi}_p\hat{\psi}_q = B, \tag{56}$$

where $B = a\sqrt{1-c^2}$. Introduce the analytic function $f = \hat{\psi} + i\hat{\chi}$ of the complex variable $z = p + iq$ ($\hat{\psi}$ and $\hat{\chi}$ are conjugate harmonic functions). Since $f'(z) = \hat{\psi}_p - i\hat{\psi}_q$ and $f''(z) = \hat{\psi}_{pp} - i\hat{\psi}_{pq}$, an easy calculation shows that we must determine an analytic function f such that

$$f''(z) - \frac{1}{2}f'(z)^2 = A - iB \tag{57}$$

with A arbitrary real constant. Setting then $f(z) = -2\log F(z)$ yields for F the linear second order equation with constant coefficients

$$F''(z) - C^2 F(z) = 0, \quad C^2 = -\frac{1}{2}(A - iB), \tag{58}$$

which obviously can be solved in terms of elementary functions. With $C = C_1 + iC_2$, and C_1, C_2 determined by $C_1^2 - C_2^2 = -A/2,\ 4C_1C_2 = B$, the solution may be written as

$$F = P\exp\left[C_1 p - C_2 q + i(C_2 p + C_1 q)\right] + Q\exp\left[-C_1 p + C_2 q - i(C_2 p + C_1 q)\right], \tag{59}$$

where P and Q are complex integration constants. Then, a wide class of exact solutions is obtained starting from $\hat{\psi} = \mathrm{Re}(-2\log F)$, depending on parameters $P_1, P_2, Q_1, Q_2, A, a, c$.

As an alternative procedure, we can express a wide class of solutions to the second equation in (53) in terms of Fourier–Laplace integral

$$w(x, y, t) = \mathrm{e}^{-\varphi} = \int_{\mathcal{C}_1\times\mathcal{C}_2} F(\alpha, \beta)\exp(\alpha x + \beta y + \alpha\beta t)\,\mathrm{d}\alpha\,\mathrm{d}\beta, \tag{60}$$

where the integral is taken along certain curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in complex planes α and β. The first equation (53) for $w(x, y, t)$

$$\left(w_{xx}w - w_x^2\right) + \left(w_{yy}w - w_y^2\right) - \left(w_{tt}w - w_t^2\right) = 0 \tag{61}$$

leads to the equality

$$\int_{\mathcal{C}_1\times\mathcal{C}_2} \mathrm{d}\alpha_1\,\mathrm{d}\beta_1 \int_{\mathcal{C}_1\times\mathcal{C}_2} F(\alpha_1, \beta_1)F(\alpha_2, \beta_2)\left[(\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2 - (\alpha_1\beta_1 - \alpha_2\beta_2)^2\right]$$
$$\times\ \exp\left[(\alpha_1 + \alpha_2)x + (\beta_1 + \beta_2)y + (\alpha_1\beta_1 + \alpha_2\beta_2)t\right]\mathrm{d}\alpha_2\,\mathrm{d}\beta_2 = 0.$$

In particular, from an ansatz like

$$F(\alpha, \beta) = \sum_{j=1}^{n} F_j\,\delta(\alpha - \alpha_j)\delta(\beta - \beta_j)$$

with arbitrary constants $F_1, \ldots, F_n$, we would obtain a set of algebraic equations for the unknown knots α_j and β_j $(j = 1, \ldots, n)$:

$$(\alpha_j - \alpha_k)^2 + (\beta_j - \beta_k)^2 = (\alpha_j\beta_j - \alpha_k\beta_k)^2, \quad j, k = 1, \ldots, n.$$

Since the case $j = k$ is trivial, we have in this way $n(n-1)/2$ algebraic relations for $2n$ unknown complex numbers. Thus, for sufficiently small $n = 2, 3, \ldots$ one can find nontrivial particular solutions.

At this point it is clear that there are some similarities in our approach with Cornille's papers [3,4]. What is important, however, is that the multiparametric solutions and the equations for parameters appear not as a consequence of a very strong a priori assumption [3,4] on the form of the solution, but as a result of the procedure following in a natural way from potentiality.

7. Conclusions and comments

It is well known that many of the existing discrete models have non-physical properties caused by spurious collision invariants [7]. At first sight, this does not apply to the 4-velocity plane Broadwell model, since the number of collision invariants is correct in this case. Of course, the lack of Galilean invariance and the absence of velocities with different magnitude restrict the number of physical problems for which this model is appropriate. However, in one space dimension (no y or z dependence), the behavior of solutions of the model does not contradict qualitatively the behavior of solutions of the Boltzmann equation.

The situation changes if we consider the Broadwell model in higher dimensions, as shown explicitly and quantitatively in the present paper by the plane case. In fact, we have solved analytically the problem of time dependent potential flows with steady limiting conditions and isotropic pressure tensor at infinity, and expressed all solutions in terms of the diagonal coordinates $x \pm y$, appearing everywhere in the classes φ_n listed in Section 5. Some of them have been recognized to have no physical meaning, since they would represent waves traveling at a speed greater than the particle speed. The most physical classes are φ_3 (with $\beta > 0$) and φ_5, which are again waves along one of the diagonals, with a propagation speed within the correct range. They differ only in the shape of the wave. The class φ_5 corresponds to a shape that is either a periodic circular function (on one diagonal) or a non-periodic hyperbolic function (on the other), which is either standing or moving at a velocity $(1-\theta^2)/(1+\theta^2) \in (-1,1)$. Solutions of this type were first discovered, by an ad hoc assumption, by Cornille [3,4], who however did not discuss the question of their unphysical features. As regards the other considered class, φ_3, the situation changes only in that the shape is instead quadratic on both diagonals. These solutions instead, to the best of the authors' knowledge, were never discovered before.

However, even these more physical classes provide examples, though less trivial, of unphysical solutions, since they represent nondissipative modes, that are not compatible with kinetic theory. Indeed these solutions, contrary to φ_1, are actually affected by collisions, in the sense that the collision term $f_2 f_4 - f_1 f_3$ is different from zero, but the waves remain undamped, and no relaxation occurs. That would be obviously impossible for any solution of the Boltzmann equation relevant to the considered problem; unfortunately all solutions to the corresponding Broadwell model equations exhibit this unphysical feature. Having determined all solutions of the discrete model, we may conclude its inadequacy in dealing with the considered flows.

The conclusion is of course not unexpected, since anyone would feel that such a caricature of the actual complicated collision term is unlikely to describe precisely physical reality. Our results show that there are wide classes of positive solutions with properties qualitatively different from the ones of the Boltzmann equation.

One of the reason for this failure is the fact that the Broadwell model is not isotropic. If we consider a class of solutions of (1) which is symmetric with respect to the diagonals $x = \pm y$, then we have to assume that $f_1 = f_2$, $f_3 = f_4$, or $f_1 = f_4$, $f_2 = f_3$. In both cases we obtain $f_2 f_4 - f_1 f_3 = 0$, i.e. there is no collisionality (thus no dissipation) for such solutions, as it was clearly indicated by Cercignani [9]. Just this property of the model is the reason for the class of solutions φ_1 described in Section 5. Another simple non-physical property is that collisions between particles with orthogonal velocity vectors do not play any role (they are trivial collisions, in the sense that gain terms cancel out exactly loss terms). As a result, we have a wide class of other collisionless solutions of the type $f_1 = F_1(x-t, y)$, $f_2 = F_2(x, y-t)$, $f_3 = f_4 = 0$, etc.

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